1. [8 marks] (R problem) Generate a realization of the AR(2) process

$$X_t = 0.75X_{t-1} - 0.5X_{t-2} + Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2)$$

of length 500 and using `spectrum()`, estimate the periodogram of the series. Using the equation from your text (and class), determine the true spectral density and overplot it **on the same plot** as the spectrum using a different colour. Be sure to hand in your code for doing all of this, as well as a clean plot.

2. [10 marks] Given $\{Z_t\}$ an orthogonal increment process, and

$$D_N(f) = \frac{\sin(N\pi f)}{N\sin(\pi f)}$$

the Dirichlet kernel, with given relationship

$$\sum_{t=1}^{N} e^{i2\pi ft} = \begin{cases} Ne^{i(N+1)\pi f} D_N(f) & f \neq 0, \pm 1, \pm 2, \ldots \\ N & f = 0, \pm 1, \pm 2, \ldots \end{cases}$$

show that

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t e^{-i2\pi ft} = \sqrt{N} \int_{-1/2}^{1/2} e^{i(N+1)\pi(f'-f)} D_N(f'-f) dZ(f')$$

with $|f| \leq 1/2$. You should use the (process) spectral representation theorem.

3. [6 marks] Show that

$$\int_{-\pi}^{\pi} e^{i(h-k)\lambda} d\lambda = \begin{cases} 2\pi & \text{, if } k = h \\ 0 & \text{, else} \end{cases}$$

(this is Question 4.1 in your text).

4. [16 marks] Suppose that $\{X_t\}$ is the noncausal and noninvertible ARMA(1,1) process satisfying

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2),$$

where $|\phi| > 1$ and $|\theta| > 1$. Define $\widetilde{\phi}(B) = 1 - \frac{1}{\phi}B$ and $\widetilde{\theta}(B) = 1 + \frac{1}{\theta}B$, and let $\{W_t\}$ be the process given by

$$W_t := \widetilde{\theta}^{-1}(B)\widetilde{\phi}(B)X_t.$$

a) Show that $\{W_t\}$ has a constant spectral density function.

b) Conclude that $\{W_t\} \sim \text{WN}(0, \sigma_W^2)$. Give an explicit formula for σ_W^2 in terms of ϕ, θ, and σ^2.

c) Deduce that $\widetilde{\phi}(B)X_t = \widetilde{\theta}(B)W_t$, so that $\{X_t\}$ is a causal and invertible ARMA(1,1) process relative to the white noise sequence $\{W_t\}$.

(for graduate students, or undergraduates seeking extra credit)

5. [20 marks] Using `mvfft()` in `R` (or equivalent in MATLAB), implement a **direct spectral estimator**. Your routine should have support for zero-padding and tapering, and should also generate

a suitable plot, modifiable by logical parameter passing. For the plotting, you should follow the pattern set by your environment: in `R`, this means presenting a plot similar in form to the `spectrum()` routine, which actually has a default `spec` object and associated plotting routine. Feel free to fall through to this routine.

You should test your routine against the setup from Question 1, presenting an example of both a periodogram estimate (i.e. taper= $1/\sqrt{N}$), and a tapered spectral estimator. The taper example can be chosen at your discretion.