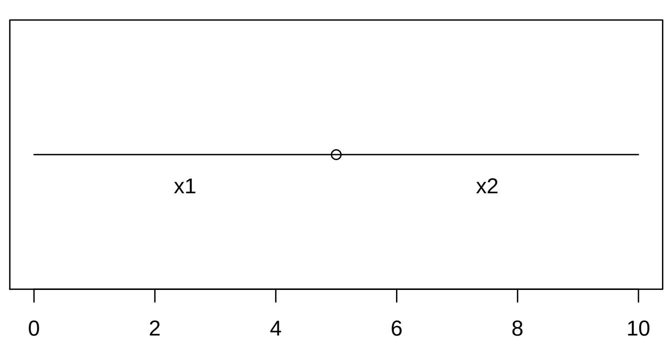
Optimization Algorithm

---

1. Draw a clearly labeled diagram representing the quantity you are interested in optimizing, as well as the relationship between the inputs.
2. Determine if there is a separate constraint equation or if the constraint has been "baked in to" your diagram. If the former, write down the constraint equation.
3. Write the fundamental optimizing equation for your problem.
4. If you have both optimizing and constraint equations, substitute for one of the variables in the optimizing equation to get a function of **only** one variable.
5. Find the critical points of the optimizing equation.
   (a) Find the derivative with respect to the single variable of the optimizing equation.
   (b) Set this equation equal to zero.
   (c) Solve for all solutions, label these as c_1, c_2, $\ldots$, where c stands for "critical point".
6. Perform the first or second derivative test for **each** critical point to classify them as minimums, maximums, or neither. These are **local** only.
7. Justify why a particular local minimum/maximum is actually a global minimum/maximum, using either the boundary values method or the theorem from class (Thursday, Jan. 24). These two methods can be reviewed in your text on pages 278 (The Closed Interval Method) and 328 (First Derivative Test for Absolute Extreme Values)

Example Problem: A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut so that the total area enclosed is a minimum? What about if the area is to be a maximum?

Solution: begin by drawing a picture:



Now, write the constraint equation from the picture as:

$$10 = x_1 + x_2.$$

Now, consider our optimizing equation. We want to consider the surface area enclosed by the wire from both a square and a circle. The area of a square is length times width (or width squared, since square), and the area of a circle is πr^2. It doesn't matter which x is assigned to each one, so let x_1 length of wire be the square, and x_2 length of wire be the circle. Then:

$$A(x_1, x_2) = \left(\frac{x_1}{4}\right)^2 + \pi\left(\frac{x_2}{2\pi}\right)^2$$

where x_1 is bent into a square, so each side has length $x_1/4$, and x_2 is bent into a circle, so $2\pi r = x_2$ (the circumferance), giving $r = \dfrac{x_2}{2\pi}$. Now, substitute in our relationship, $x_2 = 10 - x_1$, giving

$$A(x_1) = \frac{x_1^2}{16} + \pi\frac{(10 - x_1)^2}{4\pi^2} = \frac{x_1^2}{16} + \frac{(10 - x_1)^2}{4\pi}.$$

We now differentiate this equation with respect to x_1:

$$A'(x_1) = \frac{2x_1}{16} + \frac{2(10 - x_1)(-1)}{4\pi}$$

and set it equal to zero:

$$0 = \frac{x_1}{8} + \frac{x_1 - 10}{2\pi}$$

Then, solving this, we have

$$\frac{10}{2\pi} = x_1\left[\frac{1}{8} + \frac{1}{2\pi}\right]$$

or (using our calculators) $x_1 = 5.60$.

At this point, we have found a **single** critical point at $c_1 = 5.60$. We do not know if it is a minimum or maximum, and we do not know anything about the global properties of it. Start by performing the first derivative test:

x	$x < 5.60$	$x = 5.60$	$x > 5.60$
A'(x)	$-$	0	$+$

where we use $x = 5.5$ and $x = 5.7$ as sample values for evaluating $A'(x)$. Note that you could also fill in these $+/-$ values by considering

$$A'(x) = x\left[\frac{1}{8} + \frac{2}{4\pi}\right] - \frac{20}{4\pi}$$

which, upon simplification, is

$$A'(x) = 0.284x - 1.592.$$

As this is clearly linear, we can trivially fill in the 1$^{\text{st}}$ derivative table. In conclusion, since the 1$^{\text{st}}$ derivative test shows a transition from negative to positive at $c_1 = 5.60$, this critical point represents a **local minimum**.

To determine if this is a global minimum, we can apply either of the two techniques we reviewed in class (and mentioned above in the algorithm). Since we have such a simple derivative function, in this case we will make the argument that **since** we have only one critical point **and** the derivative function is always negative for x_1 values less than 5.60 **and** the derivative function is always positive for x_1 values greater than 5.60, the function $A(x_1)$ has an absolute minimum at $x_1 = 5.60$. At this point, we are done the finding of the minimum total area.

To find the maximum total area, realize that we have no critical points which are local maximums, so apply the Closed Interval Method and check the endpoints:

$$\begin{aligned} A(x_1 = 0.0) &= 7.96 \\ A(x_1 = 5.60) &= 3.50 \\ A(x_1 = 10.0) &= 6.25 \end{aligned}$$

and thus the global maximum area occurs when $x_1 = 0$, which forces all of the wire to be used in the creation of the circle. This makes sense, as the most efficient shape for enclosing area is that of the circle. Note that even though we have no local maximums in the domain $x_1 \in [0, 10]$, we still have a global maximum for $x_1 = 0$.