QUEEN'S UNIVERSITY
APSC 171J – Assignment 4: Solutions
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Due: February 7, 2013

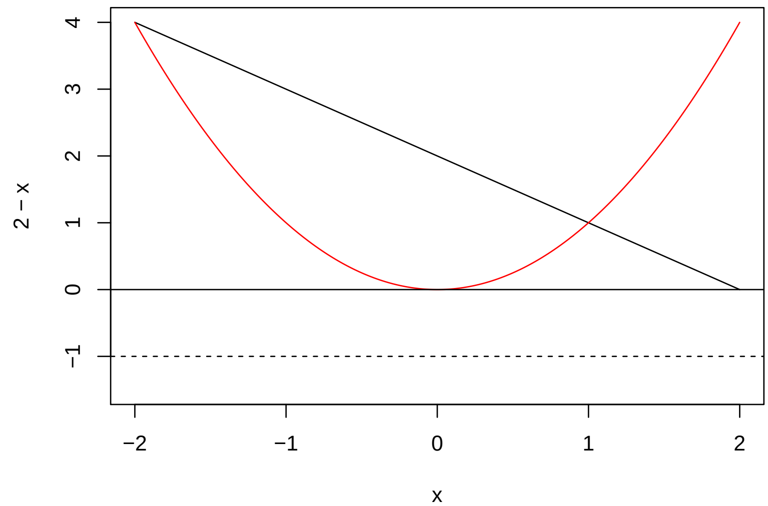
INSTRUCTIONS

- This assignment is due in-class (4:30-5:20pm) Thursday, February 7, 2013.
- Answer all questions, writing clearly on the sheets provided. **You must print this file and hand in a carefully stapled copy!** Unstapled assignments will not be accepted.
- One mark in each question is for **complete** (and mostly correct) work shown
- The second mark is for a **fully** correct solution, which **must** be placed in the box provided
- If more than two marks are provided for a question, expect the question to require more work and logically divide into sections, each of which will be worth a mark.
- Whenever possible, simplify your solution.
- There are no part marks: you will receive integer marks only for each question.

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	3	
2	2	
3	2	
4	2	
5	3	
6	3	
7	2	
8	4	
9	2	
TOTAL	23	

1. [3 marks] Find the volume of the solid generated by rotating about the line $y = -1$ the region bounded by the graphs $y = x^2$ and $y = 2 - x$. Include a carefully drawn diagram.

Begin by drawing your diagram:



Now, since the axis of rotation is horizontal, we check first with the parallel line test – but it fails! A line drawn near the bottom of the area will enter and exit on the red curve, but a line drawn near the top will enter on the red and exit on the black. Thus, we cannot use shells for this problem (or at least, not easily). So, we check the perpendicular line test, and it passes: the line always enters the area via the red curve and exits via the black curve. Thus, we can use slices.

To use slices, we need the inner and out radius to be defined.

$$r_{out} = 2 - x + 1$$
$$r_{inn} = x^2 + 1$$

where we define the y distance from the axis to the curve with $+1$ because the y values from the functions (e.g., $y = x^2$) are defined relative to the $x-$axis as zero, not the axis of rotation as given. Now, use the standard formula:

$$\begin{aligned} V &= \pi \int_{x=-2}^{1} \left[(2 - x + 1)^2 - \left(x^2 + 1\right)^2 \right] dx \\ &= \pi \int_{x=-2}^{1} \left(9 - 6x + x^2 - (x^4 + 2x^2 + 1)\right) dx \\ &= \pi \int_{x=-2}^{1} \left(8 - 6x - x^2 - x^4\right) dx \end{aligned}$$

$$
\begin{aligned}
V &= \pi \int_{x=-2}^{1} \left(8 - 6x - x^2 - x^4\right) dx \\
&= \pi \left[8x - 3x^2 - \frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{x=-2}^{1} \\
&= \pi \left[8(1) - 3(1) - \frac{1}{3} - \frac{1}{5} - \left(8(-2) - 3(-2)^2 - \frac{1}{3}(-2)^3 - \frac{1}{5}(-2)^5\right)\right] \\
&= \pi \left[5 - \frac{8}{15} + 16 + 12 - \frac{8}{3} - \frac{32}{5}\right] \\
&= \frac{117}{5}\pi \approx 73.51.
\end{aligned}
$$

Final Answer:

$\boxed{= \frac{117}{5}\pi \approx 73.51.}$

2. [2 marks] Find the solution to the integral

$$\int \sin(x)\cos(x)e^{\sin(x)}dx.$$

Let $w = \sin(x)$, then we have $\frac{dw}{dx} = \cos(x)$ so $dx = \frac{dw}{\cos(x)}$. Substituting:

$$\int w\cos(x)e^{w}\frac{dw}{\cos(x)} = \int we^{w}dw.$$

Then use Integration by Parts:

$$\begin{aligned} u &= w & v &= e^{w} \\ du &= dw & dv &= e^{w}dw \end{aligned}$$

which gives

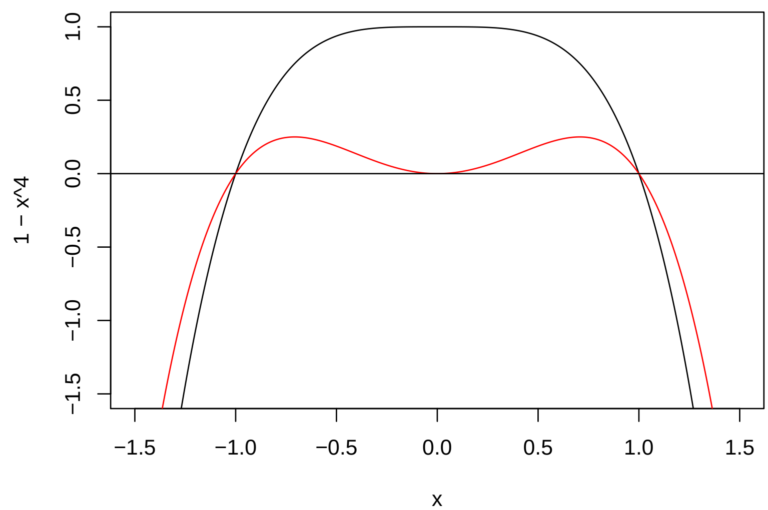
$$\begin{aligned} \int \sin(x)\cos(x)e^{\sin(x)} &= we^{w} - \int e^{w}dw = we^{w} - e^{w} + C \\ &= \sin(x)e^{\sin(x)} - e^{\sin(x)} + C \\ &= e^{\sin(x)}\left[\sin(x) - 1\right] + C. \end{aligned}$$

Final Answer:

$$\boxed{\int \sin(x)\cos(x)e^{\sin(x)}dx = e^{\sin(x)}\left[\sin(x) - 1\right] + C.}$$

3. [2 marks] Let A be the area bounded by $y = 1 - x^4$ and $y = x^2 - x^4$. Find the area of A and include a carefully labeled diagram.

Begin by drawing a clear picture:



and realize that $1 - x^4$ is greater than $x^2 - x^4$ for the enclosed region. We need the intersect points (they may be clear from the picture, but you would need them to actually *draw* the picture!), so set the two functions equal and solve:

$$1 - x^4 = x^2 - x^4 \qquad \Rightarrow \qquad x^2 = 1$$

so the intersection points are $x = \pm 1$ and $y = 0$ (for both). Now, write the equation for the area of this shape:

$$\begin{aligned}\int_{-1}^{1} \left[(1 - x^4) - (x^2 - x^4)\right] dx &= \int_{-1}^{1} \left[1 - x^2\right] dx \\ &= \left[x - \frac{1}{3}x^3\right]_{x=-1}^{x=1} = \left[1 - \frac{1}{3}\right] - \left[-1 - \frac{1}{3}(-1)^3\right] \\ &= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.\end{aligned}$$

Final Answer:

The area of the shape is $4/3$.

4. [2 marks] Determine the following definite integral:

$$\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}dx.$$

This is a Substitution integral. Let $u = 1 - x^2$, then

$$\frac{du}{dx} = -2x \qquad \Rightarrow \qquad dx = -\frac{du}{2x}$$

so

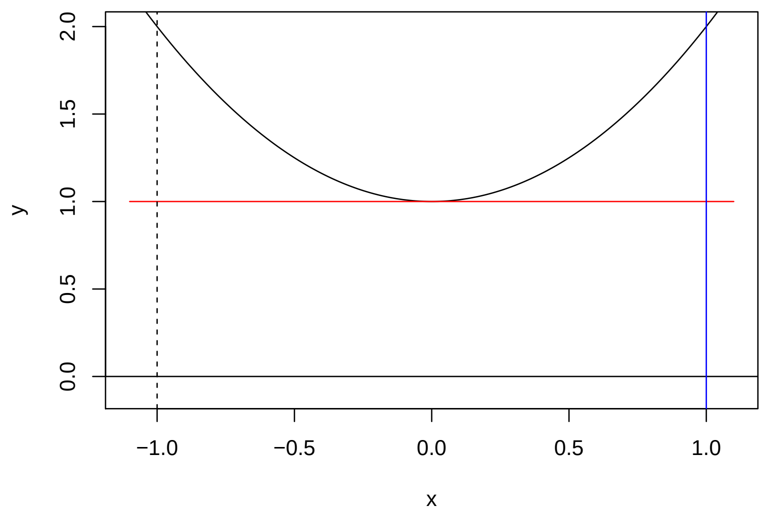
$$\begin{aligned}
\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}dx &= -\int_{x=0}^{x=0.5} \frac{x}{\sqrt{u}}\frac{du}{2x} \\
&= -\int_{x=0}^{x=0.5} \frac{1}{2}u^{-1/2}du \\
&= -\left[u^{1/2}\right]_{x=0}^{x=0.5} \\
&= -\left[\sqrt{1-x^2}\right]_{x=0}^{x=0.5} \\
&= -\left[\sqrt{1-0.5^2} - \sqrt{1-0}\right] \\
&= 1 - \sqrt{0.75} \approx 0.1340.
\end{aligned}$$

Final Answer:

$$\boxed{\int_0^{0.5} \frac{x}{\sqrt{1-x^2}}dx = 0.1340.}$$

5. [3 marks] Find the volume of the solid formed when the region bounded by $y = x^2 + 1$, $x = 1$ and $y = 1$ is rotated about the line $x = -1$. Include a carefully drawn diagram.

Begin by sketching a careful graph:



and recognize that the shape you want is only between red, blue and black curves; that is, to the right of $x = 0$. Both the vertical and horizontal line tests pass, so we can compute this volume using either method; since the question does not specify, we will use slices. A slice goes from the $y = x^2$ curve to the $x = 1$ curve, so if we are computing radii, they go from the axis of rotation at $x = -1$ to $x = \sqrt{y-1}$ and $x = 1$. Thus:

$$r_{inn} = \sqrt{y-1} + 1$$
$$R_{out} = 1 + 1 = 2$$

and thus our volume equation becomes

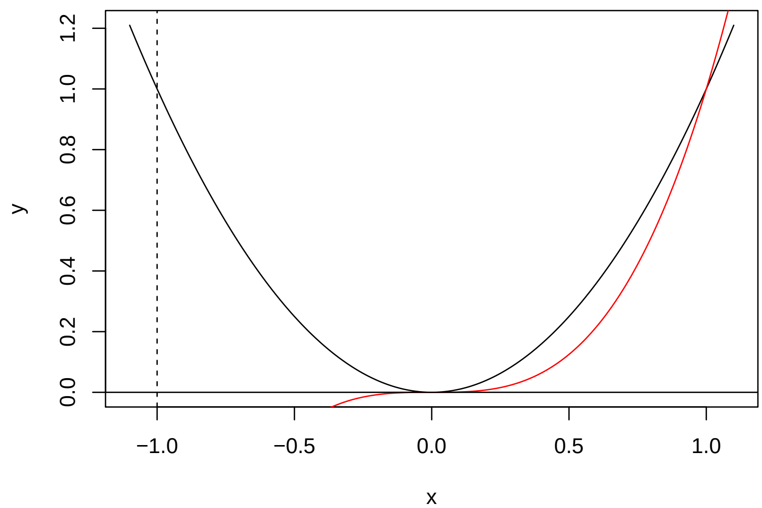
$$\begin{aligned} V &= \pi \int_1^2 \left[(2)^2 - \left(\sqrt{y-1} + 1 \right)^2 \right] dy \\ &= \pi \int_1^2 \left[4 - (y-1) - 2\sqrt{y-1} - 1 \right] dy \\ &= \pi \left[4y - \frac{1}{2}y^2 - \frac{4}{3}(y-1)^{3/2} \right]_{y=1}^{y=2} \\ &= \pi \left[4(2) - \frac{1}{2}(4) - \frac{4}{3}(1) - 4(1) + \frac{1}{2}(1) + \frac{4}{3}(0) \right] \\ &= \frac{7}{6}\pi \approx 3.665. \end{aligned}$$

Final Answer:

The volume is $7/6\pi$.

6. [3 marks] Find the volume of the solid obtained by rotating the area between the curves $y = x^2$ and $y = x^3$ about the line $x = -1$. Include a carefully drawn diagram.

Begin by sketching a careful graph:



and recognize that the shape you want is only between red and black curves; that is, between $x = 0$ and $x = 1$. Both the vertical and horizontal line tests pass, so we can compute this volume using either method; since the question does not specify, we will use shells. The shell will have height running from x^3 on the bottom to x^2 on the top, so $h = x^2 - x^3$. The radius will be $x + 1$ as the axis of rotation is offset one unit to the left. Thus our volume equation becomes

$$\begin{aligned} V &= 2\pi \int_0^1 (x+1)\left(x^2 - x^3\right) dx \\ &= 2\pi \int_0^1 \left(x^3 - x^4 + x^2 - x^3\right) dx \\ &= 2\pi \int_0^1 \left(x^2 - x^4\right) dx \\ &= 2\pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_{x=0}^{x=1} \\ &= 2\pi \left[\frac{5}{15} - \frac{3}{15} - 0 + 0\right] \\ &= \frac{4\pi}{15} \approx 0.838. \end{aligned}$$

Final Answer:

The volume is $4/15\pi$, or 0.838.

7. [2 marks] Solve the definite integral

$$\int_0^{\pi} |\cos(x)|\, dx.$$

This integral is of an absolute value, so we need to consider the behaviour of $\cos(x)$ on the region $[0, \pi]$. $\cos(x) > 0$ for $x \in (-\pi/2, \pi/2)$, so for the first half of the domain it is positive. However, $\cos(x) < 0$ for $x \in (\pi/2, 3\pi/2)$, so for the second half of the domain it is negative. Thus, write:

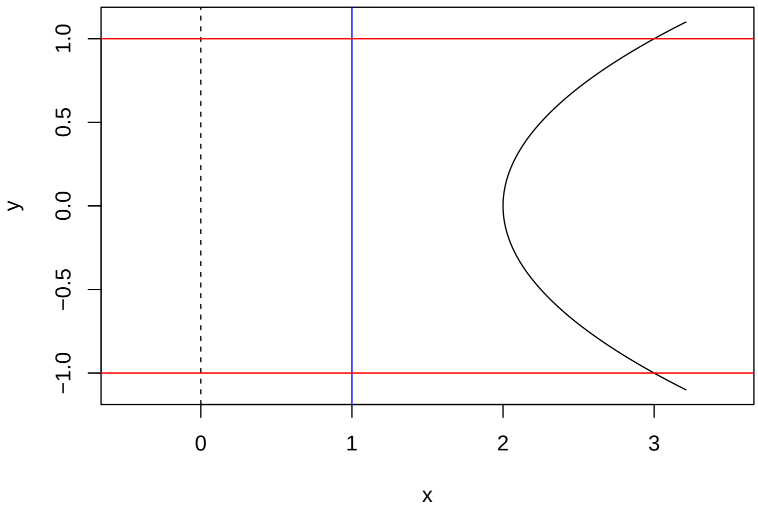
$$\begin{aligned}
\int_0^{\pi} |\cos(x)|\, dx &= \int_0^{\pi/2} \cos(x) dx + \int_{\pi/2}^{\pi} -\cos(x) dx \\
&= \sin(x)\Big|_0^{\pi/2} - \sin(x)\Big|_{\pi/2}^{\pi} \\
&= \sin(\pi/2) - \sin(0) + \sin(\pi/2) - \sin(\pi) \\
&= 1 - 0 + 1 - 0 = 2.
\end{aligned}$$

Final Answer:

The integral is equal to 2.

8. [4 marks] The region bounded by the sideways parabola $x = y^2 + 2$, the line $x = 1$ and the lines $y = -1$ and $y = 1$ is rotated about the y-axis to form a solid B. Find the volume of B. Include a carefully drawn diagram.

Begin by sketching a careful graph:



and recognize that the shape you want is between the red, blue and black curves, that is, between $x = 1$ and $x = y^2 + 2$. The vertical line test **fails** in this case (at the right side of the area), so we compute this volume using the method of slices. Each slice will have inner radius $r_{inn} = 1$ and outer radius $R_{out} = y^2 + 2$. Thus our volume equation becomes

$$\begin{aligned}
V &= \pi \int_{-1}^{1} \left[\left(y^2+2\right)^2 - (1)^2 \right] dy \\
&= \pi \int_{-1}^{1} \left[y^4 + 4y^2 + 4 - 1 \right] dy \\
&= \pi \left[\frac{1}{5}y^5 + \frac{4}{3}y^3 + 3y \right]_{y=-1}^{y=1} \\
&= \pi \left[\left(\frac{1}{5} + \frac{4}{3} + 3(1) \right) - \left(\frac{1}{5}(-1) + \frac{4}{3}(-1) + 3(-1) \right) \right] \\
&= \pi \left[\frac{136}{15} \right] \approx 28.484.
\end{aligned}$$

Final Answer:

The volume is $\frac{136}{15}\pi$ or 28.484 cubic units.

9. [2 marks] Solve the following improper integral:

$$\int_0^\infty \frac{x}{(1+x^2)^2}dx.$$

This is again a Subtitution integral, so let $u = 1 + x^2$, and then

$$\frac{du}{dx} = 2x \qquad \Rightarrow \qquad dx = \frac{du}{2x}$$

which gives

$$\begin{aligned}
\int_0^\infty \frac{x}{(1+x^2)^2}dx &= \int_{x=0}^{x=\infty} \frac{x}{u^2}\frac{du}{2x} \\
&= \int_{x=0}^{x=\infty} \frac{1}{2}u^{-2}du \\
&= \left[-\frac{1}{2}u^{-1}\right]_{x=0}^{x=\infty} \\
&= \lim_{T\to\infty}\left[-\frac{1}{2}(1+x^2)^{-1}\right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty}\left[-\frac{1}{2}(1+T^2)^{-1}+\frac{1}{2}\right] \\
&= \frac{1}{2}-\frac{1}{2}\left(\lim_{T\to\infty}\frac{1}{1+T^2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Final Answer:

$$\boxed{\int_0^\infty \frac{x}{(1+x^2)^2}dx = \frac{1}{2}.}$$