QUEEN'S UNIVERSITY
APSC 171J – Assignment 2 Solutions
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Due: January 24, 2013

INSTRUCTIONS

- This assignment is due in-class (4:30-5:20pm) Thursday, January 24
- Answer all questions, writing clearly on the sheets provided. **You must print this file and hand in a carefully stapled copy!** Unstapled assignments will not be accepted.
- One mark in each question is for **complete** (and mostly correct) work shown
- The second mark is for a **fully** correct solution, which **must** be placed in the box provided
- Whenever possible, simplify your solution.
- There are no part marks: you will receive 0, 1 or 2 on each question.

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
TOTAL	20	

1. [2 marks] Find $\frac{dy}{dx}$ if

$$y(x) = \int_0^{\ln(x)} e^{\sin(t)} dt.$$

We apply the Leibniz Integral Rule:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x).$$

This gives

$$\begin{aligned} \frac{d}{dx} \left(\int_0^{\ln(x)} e^{\sin(t)} dt \right) &= e^{\sin(\ln(x))} \cdot \frac{d}{dx} (\ln(x)) - e^{\sin(0)} \cdot 0 \\ &= \frac{e^{\sin(\ln(x))}}{x}. \end{aligned}$$

Final Answer:

$$\boxed{\frac{d}{dx} \left(\int_0^{\ln(x)} e^{\sin(t)} dt \right) = \frac{e^{\sin(\ln(x))}}{x}}$$

2. [2 marks] Find $\frac{dy}{dx}$ if

$$y(x) = (1+x^2)^{\sin(x)}.$$

Begin by taking the natural logarithm of both sides:

$$\begin{aligned}
\ln(y(x)) &= \ln\left[(1+x^2)^{\sin(x)}\right] \\
\ln(y(x)) &= \sin(x)\ln\left[(1+x^2)\right] \text{ (and take the derivative on both sides)} \\
\frac{1}{y}y' &= \cos(x)\ln\left[1+x^2\right] + \sin(x)\frac{1}{1+x^2}\cdot 2x \\
y'(x) &= y\left[\cos(x)\ln\left[1+x^2\right] + \sin(x)\frac{2x}{1+x^2}\right] \\
y'(x) &= (1+x^2)^{\sin(x)}\left[\cos(x)\ln(1+x^2) + \frac{2x\sin(x)}{1+x^2}\right].
\end{aligned}$$

Final Answer:

$$\boxed{y'(x) = (1+x^2)^{\sin(x)}\left[\cos(x)\ln(1+x^2) + \frac{2x\sin(x)}{1+x^2}\right]}$$

3. [2 marks] Find $\frac{dy}{dx}$ if

$$xy^2 - x^2y = 6$$

and evaluate the derivative at $(x, y) = (2, 3)$.

Using implicit differentiation:

$$\begin{aligned} y^2 + x \cdot 2yy' - 2xy - x^2y' &= 0 \\ y^2 - 2xy + 2xyy' - x^2y' &= 0 \\ y'\left[2xy - x^2\right] &= 2xy - y^2 \\ y'(x) &= \frac{2xy - y^2}{2xy - x^2} \end{aligned}$$

Then substitute:

$$y'(2,3) = \frac{2(2)(3) - (3)^2}{2(2)(3) - (2)^2} = \frac{3}{8}.$$

Final Answer:

$$y'(x) = \frac{2xy - y^2}{2xy - x^2} \text{ and } y'(2,3) = 3/8$$

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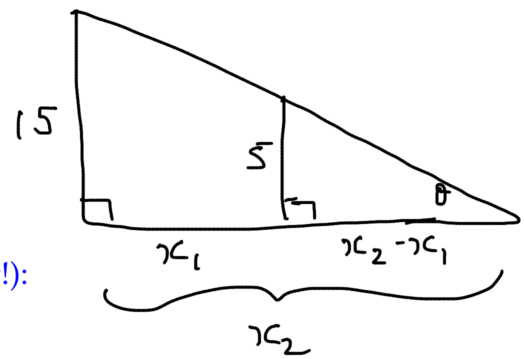
4. [2 marks] A student who is 5′ (feet) tall is walking down a sidewalk away from a lamppost which is 15′ (feet) tall. If she is moving at 2 feet/second when she is 10′ (feet) from the base of the lamppost, how fast is the tip of her shadow moving (where the tip of her shadow represents her 'head').

Using similar triangles with x_1 being the girl's distance from the lamppost and x_2 being the tip of the shadow's distance from the lamppost (see diagram):

$$\begin{aligned} \frac{15}{x_2} &= \frac{5}{x_2 - x_1} \\ 15x_2 - 15x_1 &= 5x_2 \\ 10x_2 &= 15x_1 \\ x_2 &= 1.5x_1 \end{aligned}$$



Now, apply implicit differentiation to this (**before** substituting!):

$$x_2' = 1.5x_1'$$

and substitute $x_1' = 2$ ft/second:

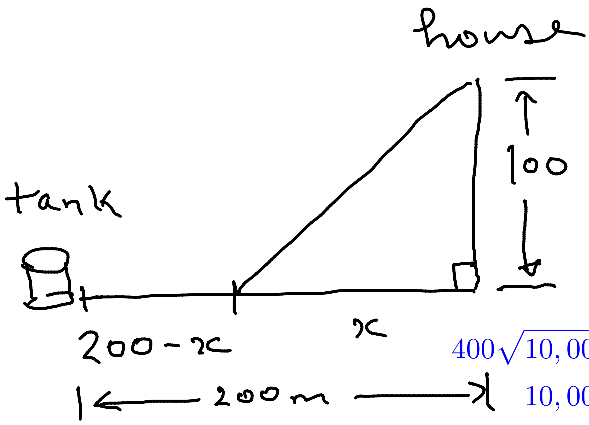
$$x_2' = 3\text{ft/sec.}$$

Final Answer:

$$\boxed{x_2' = 3\text{ft/second}}$$

5. [2 marks] A gas pipeline is to be constructed from a storage tank which sits at the edge of a road. This pipeline is to run to a house which is 200m down the road and 100m back from the road (perpendicularly). Pipe laid along the road costs $\$400$ / meter, while pipe laid off-road costs $\$500$ / meter. Find the minimum cost of laying the pipeline.

Examining the diagram, the length of the pipeline laid on the road is $200 - x$, which leaves a off-road section equal to the hypotenuse of a triangle with sides x and 100. Thus:



$$\begin{aligned} C(x) &= 400 \cdot (200 - x) + 500 \cdot \sqrt{100^2 + x^2} \\ &= 80,000 - 400x + 500\sqrt{10,000 + x^2} \end{aligned}$$

and differentiate to find critical points:

$$\begin{aligned} C'(x) &= -400 + 500\frac{1}{2}\left(10,000 + x^2\right)^{-1/2} \cdot 2x = 0 \\ 400 &= \frac{500x}{\sqrt{10,000 + x^2}} \\ 400\sqrt{10,000 + x^2} &= 500x \\ 10,000 + x^2 &= 1.5625x^2 \\ x^2 &= \frac{10,000}{0.5625} \\ x &= 133.333\text{m}. \end{aligned}$$

Thus, we have a critical point at $x = 133.333$. Now, check to see that this point is a minimum for the cost function by using the first derivative test: where we have used $x = 133$ and $x = 134$ to plug into

x	$x < 133.33$	$x = 133.33$	$x > 133.33$
C′(x)	$-$	0	$+$

$C'(x)$ to check these values. Now, since there is only one critical point, and it is a relative minimum, the slopes to the left and right of the critical point cannot change on the domain $x \in [0, 200]$. This implies that this relative minimum is also a global minimum, so our minimum cost for the pipline occurs at $x = 133.33$m and is

$$\begin{aligned} C(133.33) &= 80,000 - 400(133.33) + 500\sqrt{10,000 + (133.33)^2} \\ &= \$110,000.00. \end{aligned}$$

Final Answer:

$x = 133.33$m gives a min of $C(133.33) = \$110,000.$

6. [2 marks] Find all local minima and maxima of the function

$$y(x) = 2x^3 - 9x^2 + 12x - 3.$$

To find local minima and maxima (and classify them), take the first derivative and set it equal to zero:

$$\begin{aligned} y'(x) &= 2(3)x^2 - 9(2)x + 12 = 0 \\ 0 &= 6x^2 - 18x + 12 \\ 0 &= x^2 - 3x + 2 \\ 0 &= (x-2)(x-1) \end{aligned}$$

which has solutions $x = 1$ and $x = 2$. These are critical points, so use the first derivative test to classify them: Thus the critical point at $x = 1$ is a local maximum, and the critical

x	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
C′(x)	$+$	0	$-$	0	$+$

point at $x = 2$ is a local minimum. The first derivative test table was filled out by using $y'(x) = (x-2)(x-1)$ and examining the signs of the components for arbitrary inputs in those ranges.

Final Answer:

$x = 1$ is a local maximum ; $x = 2$ is a local minimum.

7. [2 marks] Find $\frac{dy}{dx}$ for

$$y(x) = 2 + \int_1^{\cos(x)} \sqrt{1-t^2}dt.$$

We again apply the Leibniz Integral Rule. Note that $d/dx(2) = 0$ so this piece will drop out.

$$\begin{aligned}
y'(x) &= \frac{d}{dx}\int_1^{\cos(x)} \sqrt{1-t^2}dt \\
&= \sqrt{1-\cos^2(x)} \cdot \frac{d}{dx}\cos(x) - \sqrt{1-1^2} \cdot 0 \\
&= -\sqrt{\sin^2(x)}\sin(x) \\
&= -\sin^2(x).
\end{aligned}$$

Final Answer:

$$\boxed{y'(x) = -\sin^2(x).}$$

8. [2 marks] Find $\frac{dy}{dx}$ for
$$\arctan(x+y) = xy.$$

$$\begin{aligned}
\frac{1}{1+(x+y)^2} \cdot \frac{d}{dx}(x+y) &= x'y + xy' \\
\frac{1}{1+(x+y)^2} \cdot (1+y') &= y + xy' \\
1 + y' &= (y + xy')\left(1+(x+y)^2\right) \\
y' - xy'\left(1+(x+y)^2\right) &= y\left(1+(x+y)^2\right) - 1 \\
y'\left[1 - x\left(1+(x+y)^2\right)\right] &= y\left(1+(x+y)^2\right) - 1 \\
y' &= \frac{y\left(1+(x+y)^2\right)-1}{1-x\left(1+(x+y)^2\right)}.
\end{aligned}$$

You could try to simplify this further, but it really doesn't get any nicer.

Final Answer:

$$\boxed{y' = \frac{y\left(1+(x+y)^2\right)-1}{1-x\left(1+(x+y)^2\right)}.}$$

9. [2 marks] Find $\frac{dy}{dx}$ for

$$y(x) = x^{1/e^x} = x^{e^{-x}}.$$

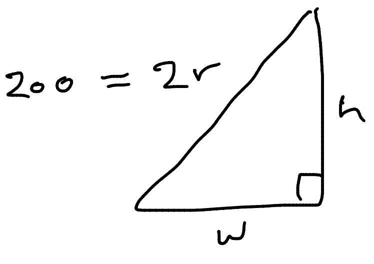
This is again the "natural log both sides" trick.

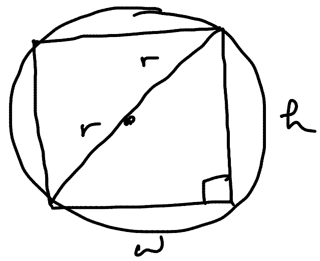
$$\begin{aligned}
\ln(y(x)) &= \ln\left[x^{e^{-x}}\right] \\
\frac{d}{dx}\left[\ln(y(x))\right] &= \frac{d}{dx}\left(e^{-x}\ln[x]\right) \\
\frac{1}{y}y' &= -e^{-x}\ln(x) + e^{-x}\frac{1}{x} \\
y'(x) &= y\left[\frac{e^{-x}}{x} - e^{-x}\ln(x)\right] \\
&= x^{e^{-x}}\left[\frac{e^{-x}}{x} - e^{-x}\ln(x)\right].
\end{aligned}$$

Final Answer:

$$\boxed{y'(x) = x^{e^{-x}}\left[\frac{e^{-x}}{x} - e^{-x}\ln(x)\right].}$$

10. [2 marks] A rectangular beam is cut from a cylindrical log with radius 100 cm. The strength of the beam is proportional to the product of the width w and the square of the height h of the cross-section. Find the dimensions of the strongest beam that can be cut from the given log.





Recognize from the diagram that the width and height of the cross-section of the beam form a right triangle with hypotenuse equal to $2r$, for $r = 100$. Thus, the constraint equation becomes

$$w^2 + h^2 = 200^2.$$

Then, letting S be the true strength, and R the replacement equation which ignores the constant of proportionality:

$$\begin{aligned}
S &\propto wh^2 \\
R &= w\left[200^2 - w^2\right] \\
&= 40,000w - w^3 \\
R' &= 40,000 - 3w^2 = 0 \\
40,000 &= 3w^2 \\
w &= 115.47\text{cm}.
\end{aligned}$$

This gives $h = \sqrt{40,000 - 40,000/3} = 163.30\text{cm}$. Now, check that this critical point is a local maximum: and since there is only one critical point on the domain, these slopes must extend

w	$w < 115.47$	$w = 115.47$	$w > 115.47$
R′(w)	$+$	0	$-$

forever, as they cannot change (if they did, there would be another critical point where the change happened). Thus, this point is also a global maximum.

Final Answer:

Maximum strength occurs for

$$w = 115.47\text{cm and } h = 163.30\text{cm}.$$